Quantum Hall effect in quantum electrodynamics

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We consider the quantum Hall effect in quantum electrodynamics and find a deviation from the quantum-mechanical prediction for the Hall conductivity due to radiative antiscreening of electric charge in an external magnetic field. A weak dependence of the universal von Klitzing constant on the magnetic field strength, which can possibly be observed in a dedicated experiment, is predicted.

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The quantum Hall effect (QHE) (Refs. 1 and 2) is a remarkable phenomenon remaining in the focus of experimental and theoretical research over the last three decades. The study of the QHE led to development of new fundamental physical concepts.[3,4] At the same time the QHE plays a crucial role in metrology and determination of fundamental constants.[5]

In the QHE, in sharp contrast to the prediction of classical electrodynamics, the conductivity of the two-dimensional electron system in a strong transverse magnetic field at low temperature has plateaus as a function of the magnetic field strength. At these plateaus the conductivity is given by integer or specific fractional multiples of R_K^{-1}, where R_K is a universal parameter known as the von Klitzing constant. A simple quantum-mechanical consideration of the noninteracting electron gas relates it to the fine-structure constant[6]

$$R_K^{-1} = 2\alpha. \tag{1}$$

A remarkable property of a two-dimensional electron system in magnetic field is that this naive result is stable against all kinds of perturbations which do not result in a qualitative change in the Landau spectrum. This has been proven first in Ref. 7 (see also Ref. 8) by an elegant use of gauge invariance. Later, a relation of the Hall conductivity to the topological invariants of the adiabatic ground-state space has been established.[9–11] Much work has been done to find a possible deviation from Eq. (1) (see, e.g., Ref. 12). However, leaving aside finite temperature and edge effects, no universal corrections have been found and Eq. (1) is currently considered to be exact.[5] This would distinguish the quantum Hall conductance as one of a very few characteristics of many-particle interacting quantum systems exactly predicted by theory. On the other hand the exact relation (1) would allow for determination of the fine structure constant with *a priori* zero theoretical uncertainty.

The purpose of this Brief Report is to show that in quantum electrodynamics (QED) quantum field effects lead to deviation from the quantum-mechanical prediction for the Hall conductance. The physics behind this phenomenon is in a modification of the electromagnetic coupling of electrons due to vacuum polarization by highly virtual electron-positron pairs in a strong magnetic field, which can roughly be described as radiative antiscreening of the electric charge at large distance. The main result of this Brief Report, which is the leading-order QED correction to Eq. (1), is given by Eq. (17).

Following Ref. 7 we consider the Hall current I around an asymptotically large loop of a two-dimensional ribbon subjected to a time-independent locally homogeneous magnetic field B and an electric field E. The spatial vectors $\boldsymbol{I}$, $\boldsymbol{B}$, and $\boldsymbol{E}$ are orthogonal to each other and the magnetic field is normal to the ribbon surface (see Fig. 1). For further analysis it is convenient to introduce an auxiliary magnetic flux Φ through the loop. The Hall conductivity R_H^{-1} is defined by the equation $I=R_H^{-1}V$, where V is the potential drop across the ribbon. In QHE it is given by $R_H^{-1}=\nu R_K^{-1}$, where the filling factor ν can be either integer[1] or fractional.[2] We focus on the integer QHE since the case of fractional ν can be understood as the integer QHE for fractionally charged quasiparticles.[3]

The QHE is a collective phenomena in condensed matter and the analysis of quantum field effects in such a system is not straightforward. To get a systematic description of interacting electrons in QED we use the nonrelativistic effective theory approach.[13] Let us briefly outline it. The core idea of the method is to disentangle the contributions of excitations corresponding to widely separated dynamical scales. In the absence of interaction with the medium, the dynamics of an electron in a magnetic field B is characterized by three scales: the hard scale of the electron mass m, the soft scale of the cyclotron momentum $\sqrt{eB}$, and the ultrasoft scale of the cyclotron energy eB/m. If the parameter

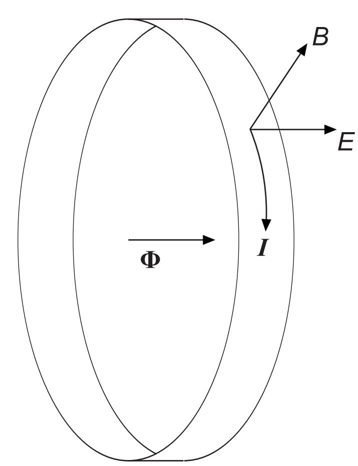

FIG. 1. Geometry of the Hall current. The size of the loop is much larger than any other scale of the problem and the magnetic field is homogeneous near the surface of the ribbon.

$$\beta = \frac{eB}{m^2} \quad (2)$$

is small, the above scales are widely separated and the effective-field theory method is applicable. Interaction with the medium results in appearance of additional soft and ultrasoft scales. However, the specific nature of these scales is not important until the above hierarchy is violated. The electrons in the ground state of the Landau spectrum are nonrelativistic and have soft momentum and ultrasoft energy. The hard and soft excitations could only appear as virtual states and the dynamics of the real electrons is determined by an effective Schrödinger equation and the multipole interaction with the ultrasoft photons.[14] The corresponding effective Hamiltonian is of the following form:

$$\mathcal{H} = e^* A_0 - \frac{\boldsymbol{D}^2}{2m^*} + \delta\mathcal{H}, \quad (3)$$

where A_0 is the potential of the electric field E, $\boldsymbol{D}$ is the spatial covariant derivative, e^* (m^*) stands for the effective charge (mass) of the electron, and $\delta\mathcal{H}$ represents the radiative and relativistic corrections as well as the interaction with the medium. The entire contribution of the hard and soft excitations is encoded in the parameters of the Hamiltonian, which can be systematically computed in QED as a series in α and β or, in general, a ratio of the scales present in the problem. The quantum Hall conductivity is known to be independent of m^* and $\delta\mathcal{H}$.[7,9] The ultrasoft contribution represents the effect of retardation and cannot be reduced to a variation of the Hamiltonian. Nevertheless, the arguments of Refs. 7 and 9 hold and the ultrasoft contribution to R_H^{-1} vanishes, which can be checked by an explicit calculation (see, e.g., Ref. 15).

Thus the only source of the corrections to Eq. (1) is electron coupling to the external fields. This coupling is modified by vacuum polarization through creation of hard virtual electron-positron pairs. In the absence of a magnetic field this effect is reabsorbed by the on-shell renormalization of the physical electron charge e. For a nonvanishing magnetic field the vacuum polarization cannot be "renormalized out" and the effective charge does differ from e. Since the magnetic field B explicitly breaks down the Lorentz invariance, the effective charges are in general different for different external fields. For the calculation of the Hall conductivity we need besides e^* another effective charge e', which parametrizes the coupling of the electrons to the vector potential of the auxiliary magnetic flux in the covariant derivative $\boldsymbol{D} = \partial - ie'\boldsymbol{A}^{\Phi} + \cdots$.

The effective charges are determined by the behavior of the vacuum polarization tensor $\Pi_{\mu\nu}(q)$ at small four-momentum transfer q. By using the integral representation of Refs. 16 and 17 it is straightforward to derive the leading variation of the polarization tensor due to the magnetic field in the limit $q \to 0$, which reads

$$\delta\Pi_{\mu\nu}(q) = -\frac{\alpha}{\pi}\beta^2 \frac{1}{45}[2(g_{\mu\nu}q^2 - q_\mu q_\nu) - 7(g_{\mu\nu}q^2 - q_\mu q_\nu)_{\parallel} + 4(g_{\mu\nu}q^2 - q_\mu q_\nu)_{\perp}]. \quad (4)$$

The correction to the polarization tensor is transverse because of the gauge invariance. At the same time the Lorentz invariance is broken and Eq. (4) includes the transverse projectors in the "parallel" $(q_0, \boldsymbol{q}_{\parallel})$ and "orthogonal" $(\boldsymbol{q}_{\perp})$ two-dimensional subspaces of the whole four-dimensional Minkowskian momentum space $(q_0, \boldsymbol{q})$. Here $\boldsymbol{q}_{\parallel}$ and $\boldsymbol{q}_{\perp}$ components correspond to the spatial momentum parallel and orthogonal to the magnetic field, respectively.

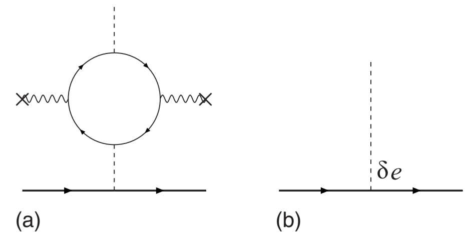

FIG. 2. Feynman diagrams (a) in full QED and (b) in the nonrelativistic effective theory representing the antiscreening of the electric charge in the external magnetic field. The arrow lines correspond to the free-electron propagators. The bold arrow lines correspond to the electron propagating in the external magnetic field. The dashed lines represent the electric potential, the crossed wavy lines represent the external magnetic field, and $\delta e = e^* - e$.

Let us now consider the effective charge e^*, which parametrizes the interaction of the electron to the homogeneous electric field. The first two terms in square brackets of Eq. (4) result in a modification of the static Coulomb potential between two pointlike charges,[18]

$$V(\boldsymbol{r}) = \frac{\alpha}{r}\left[1 + \frac{\alpha}{\pi}\beta^2\left(\frac{2}{45} - \frac{7}{90}\sin^2\theta\right)\right], \quad (5)$$

where θ is the angle between $\boldsymbol{B}$ and $\boldsymbol{r}$, i.e., the Coulomb interaction in the presence of the magnetic field becomes anisotropic. Taking an infinite uniformly charged plane as a source of E and using potential (4) for the electron interaction with the charge density, one gets the following result:

$$e^* = e\left[1 + \frac{11}{180}\frac{\alpha}{\pi}\beta^2\right]. \quad (6)$$

Thus the vacuum polarization in the magnetic field enhances the electron coupling to the electric field which generates the Hall current. Graphically the effect is represented by the Feynman diagrams in Fig. 2.

Similar effect occurs in the case of the effective charge e' although the perturbative coefficient is different. The vector potential of the auxiliary magnetic flux has only $\boldsymbol{A}_{\perp}$ component and its momentum has only $\boldsymbol{q}_{\parallel}$ component. Thus only the first term of Eq. (4) contributes to the corresponding coupling and one gets $e' = e[1 + \alpha\beta^2/(45\pi)]$. Note that e' is exactly the parameter which appears in the quantization condition for the auxiliary magnetic flux through the contour of the Hall current. Hence in the presence of the magnetic field B the "effective" flux quantum becomes $\Phi_0' = 2\pi/e'$ or

$$(\Phi_0')^{-1} = \frac{e}{2\pi}\left[1 + \frac{1}{45}\frac{\alpha}{\pi}\beta^2\right]. \quad (7)$$

Now we are in a position to derive the correction to the quantum-mechanical result for the Hall conductivity. In general, the Hall current is given by the integral of the current density over the ribbon cross section,

$$I = \int [j_I(\boldsymbol{r}) + \delta j_I(\boldsymbol{r})] dr_E dr_B, \tag{8}$$

where $\boldsymbol{r}=(r_I, r_E, r_B)$ is a vector with the components parallel to $\boldsymbol{I}$, $\boldsymbol{E}$, and $\boldsymbol{B}$, respectively. A single electron contribution to the unperturbed current density can be written as follows:

$$j_I(\boldsymbol{r}) = -i\frac{e}{m}\phi^*(\boldsymbol{r}) D_I \phi(\boldsymbol{r}), \tag{9}$$

where $\phi(\boldsymbol{r})$ is the eigenfunction of the Hamiltonian (3). In QED the perturbation to the current density due to the vacuum polarization (4) reads

$$\delta j_I(\boldsymbol{r}) = \int \left[-\frac{1}{2}\frac{\delta\Pi_{II}(\boldsymbol{q})}{\boldsymbol{q}^2} \right] \bar{j}_I(\boldsymbol{q}) e^{i\boldsymbol{r}\boldsymbol{q}} \frac{d\boldsymbol{q}}{(2\pi)^3}, \tag{10}$$

where $\bar{j}_I(\boldsymbol{q})$ is the Fourier transform of $j_I(\boldsymbol{r})$. Integrating the exponent in representation (10) over $dr_E dr_B$ gives the product of delta functions $\delta(q_E)\delta(q_B)$. Hence, the function $\delta\Pi_{II}(\boldsymbol{q})/\boldsymbol{q}^2$ under the integral over $d\boldsymbol{q}$ can be replaced with $\delta\Pi_{II}(q_I)/q_I^2$ and is reduced to a constant $-2\alpha\beta^2/(15\pi)$ up to longitudinal terms, which vanish because of the current conservation. Thus one has

$$\int \delta j_I(\boldsymbol{r}) dr_E dr_B = \frac{1}{15}\frac{\alpha}{\pi}\beta^2 \int j_I(\boldsymbol{r}) dr_E dr_B \tag{11}$$

and the expression for the Hall current takes the following form:

$$I = \left(1 + \frac{1}{15}\frac{\alpha}{\pi}\beta^2\right) \int j_I(\boldsymbol{r}) dr_E dr_B. \tag{12}$$

The integral in Eq. (12) can be expressed through the derivative of the electron energy $\mathcal{E}$ in Φ, i.e.,

$$\int j_I(\boldsymbol{r}) dr_E dr_B = -\frac{e}{e'}\frac{d\mathcal{E}}{d\Phi} \tag{13}$$

(see, e.g., Ref. 12). Thus our final expression for the Hall current reads

$$I = -C\frac{d\mathcal{E}_t}{d\Phi}, \tag{14}$$

where $\mathcal{E}_t$ is the total energy of the electrons contributing to the current and

$$C = 1 + \frac{2}{45}\frac{\alpha}{\pi}\beta^2 \tag{15}$$

is the matching coefficient. As has been shown in Ref. 7, the flux Φ acts as a quantum pump: changing it by n quanta Φ_0' results in a net transfer of $n\nu$ electrons across the ribbon, which corresponds to an energy variation of $n\nu e^* V$. Thus for the Hall conductivity one gets

$$R_H^{-1} = \nu\frac{Ce^*}{\Phi_0'}. \tag{16}$$

Note that this result does not depend on the global geometry of the Hall current. For example, in the annular geometry considered in Ref. 8 the auxiliary flux is parallel to the magnetic field $\boldsymbol{B}$ and is quantized differently. The difference, however, is compensated by the variation of the matching coefficient since Eq. (12) does not change. Putting together Eqs. (6), (7), (15), and (16) we obtain the final expression for the von Klitzing constant,

$$R_K^{-1} = 2\alpha\left[1 + \frac{23}{180}\frac{\alpha}{\pi}\beta^2\right] \tag{17}$$

or in physical units

$$R_K^{-1} = \frac{e^2}{2\pi\hbar}\left[1 + \frac{23}{180}\frac{\alpha}{\pi}\left(\frac{\hbar e B}{c^2 m^2}\right)^2\right]. \tag{18}$$

We would like to emphasize that the characteristic distance of the vacuum fluctuations resulting in the correction to the Hall conductivity is given by the electron Compton wavelength on the order of 10^{-12} m, which is far smaller than the actual thickness of the layer, where the electrons are localized, on the order of 10^{-8} m. Thus the correction to R_K is due to an intrinsically three-dimensional effect, which is not prohibited by the topological and gauge invariance arguments developed in two dimensions.

The correction term in Eq. (18) can be rewritten as follows:

$$\frac{23}{180}\frac{\alpha}{\pi}\left(\frac{B}{B_0}\right)^2, \tag{19}$$

where $B_0 = c^2 m^2/(\hbar e) \approx 4.41 \times 10^9$ T. A typical value of the magnetic field in current experiments corresponds to $B/B_0 \sim 10^{-8}$. Thus numerically Eq. (19) amounts to a tiny 10^{-20} correction. This is well beyond the available accuracy of the von Klitzing constant determination, which is about 1 part per 10^8.[5] However, this accuracy is limited mainly by the absence of an independent standard of resistance. Studying the variation of R_K with B does not have this restriction and can be performed by means of a different experimental method and with a presumably significantly higher accuracy. A renowned example of a similar phenomenon is given by the system of neutral K_S and K_L mesons, where the absolute experimental accuracy for the mass difference is about 12 orders of magnitude higher than for the average mass.[19] Thus it is an open question whether the evidence of Eq. (17) can be obtained with the available experimental facilities.

On the other hand, there is no fundamental reason which rules out the possibility of the observation of the phenomenon in a dedicated future experiment. A possible scheme of such an experiment involves two identical samples assembled in a single electric circuit and exposed to different magnetic fields. The effect becomes observable when the Hall voltage difference between the samples due the correction term in Eq. (17) reaches the resolution of the measuring device, e.g., based on the Josephson frequency-voltage conversion. Note that the voltage difference can be increased by orders of magnitude if one uses stronger magnetic field and larger values of the Hall current. We would like to emphasize that the quantum Hall conductance is topologically protected against any other type of corrections including the finite-size effects,[7,8] which otherwise would mask the tiny effect of vacuum polarization.

In summary, the leading QED correction to the quantum-mechanical result for the Hall conductivity is derived. It results in a weak dependence of the universal von Klitzing constant on the magnetic field strength. This remarkable and unexpected manifestation of a fine nonlinear quantum field effect in a collective phenomenon in condensed matter merits a dedicated experimental analysis.

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